Apr 28.

Def (M^n, g) is <u>asymptotically</u> ~~flat~~ euclidean, order $\sigma > 0$

if $\exists\ K \subset\subset M^n$ s.t.

$\exists\ \Phi : M^n \setminus K \longrightarrow \mathbb{R}^n \setminus \overline{B}_R(0)$ s.t.

$$g_{ij} = \delta_{ij} + O(r^{-\sigma}), \quad \partial^{|k|} g_{ij} = O(r^{-|k|-\sigma}).$$

Def When $\sigma > \frac{n-2}{2}$, ADM mass is well-defined.

$$m(g) = \lim_{r\to\infty} \int_{S_r} \partial_i g_{ij} - \partial_j g_{ij}\, dA^j.$$

$$R = \partial_j(\partial_i g_{ij} - \partial_j g_{ii}) + E(g) \qquad O(r^{-2\sigma'-2})$$

$\sigma' < \sigma$

Conj (PMT)

if (M^n, g) AE, $R \geq 0$, $R \in L^1$, $\sigma > \frac{n-2}{2}$, then $m(g) \geq 0$

"$=$" iff $(M^n, g) = (\mathbb{R}^n, \delta_{ij})$.

$(3 \leq n \leq 7)$ by Schoen-Yau.

Question: new proof for $n=3$ using Ricci flow.

Main idea:

- (M^n, g) AE, start Ricci flow $g(t), t \in [0, T)$
  $g(0) = g$
- if $T < \infty$, $n=3$ do Ricci flow with surgery

Preserve quantities.

- (M^n, g) complete with bounded curvature

  $\exists$ Ricci flow $g(t)$, $g(0) = g$ complete with bounded curvature on $[0, T)$ (Shi'89)

  uniqueness (Chen-Zhu'02)

① AE preserved.

② ADM mass constant.

idea: ① maximal principle

$g(t)$ on $[0, T]$, solution of Ricci flow with bounded curvature.

$$Lu = u_t - \Delta u - \langle \underbrace{X(t)}_{\text{bdd smooth}}, \nabla u \rangle - \underbrace{G(u,t)}_{\text{Lipschitz loc.}}$$

$|u(x,t)| \leq \exp(b\, d_{g_t}(O, x) + 1)$ some b.

$|u(x,0)| \leq C$

$$\Rightarrow u(x,t) \leq U(t), \quad U(t) \text{ solving } \begin{cases} \dfrac{dU}{dt} = G(U,t) \\ U(0) = C \end{cases}$$

Now (M^n, g) is AE of order $\sigma > 0$.

$\Rightarrow |Rm|(0) = O(r^{-2-\sigma})$

Want $|Rm|(t) \leq C r^{-2-\sigma}$ on $[0, T]$. (using max prin).

if so, then

$$\left|\log\left\{\frac{g(x,t)(U,U)}{g(x,0)(U,U)}\right\}\right| = \left|\int_0^t \frac{-2\,Ric(x,s)(U,U)}{g(x,s)(U,U)}\,ds\right|$$

$$\leq \int_0^t \left|-2\,Ric(x,s)\left(\frac{U}{\|U\|},\frac{U}{\|U\|}\right)\right| ds \leq Cr^{-2-\sigma}$$

$$\Rightarrow g_{ij}(t) = g_{ij}(0)\left(1+O(r^{-2-\sigma})\right)$$

$$= \left(1+O(r^{-\sigma})\right)\left(1+O(r^{-2-\sigma})\right).$$

To show $|Rm| \leq Cr^{-2-\sigma}$ for $t \in [0,T]$

$$\partial_t |Rm|^2 \leq \Delta |Rm|^2 + |16\,Rm|^3$$

$(|Rm| \leq S$ on $[0,T])$

$$\leq \Delta |Rm|^2 + 16S|Rm|^2.$$

(Take $u = e^{-16St}|Rm|^2$)

$$\partial_t u \leq \Delta u. \qquad u \sim |Rm|^2 \sim r^{-4-2\sigma}.$$

(Take $h = r^{4+2\sigma}$ and $w = uh$)

$$(\partial_t - \Delta)\,w \leq \underbrace{\left(\frac{2|\nabla u|^2 - h\Delta u}{h^2}\right)}_{B} w - 2\nabla\log h\,\nabla w.$$

wants to be bounded.

$$\partial_t |\nabla h|^2 = 2\, Ric(\nabla h, \nabla h) \le 2 \underbrace{|Rm|}_{\le S} \cdot |\nabla h|^2$$

$$\partial_t (\Delta h) = 2 \langle Ric, \nabla^2 h \rangle.$$

$$C^{-1} g(0) \le g(t) \le C g(0) \quad \text{for } t \in [0, T].$$

These give
$$|\nabla h|^2 \sim r^{6+4\sigma}$$
$$h \Delta h \sim r^{6+4\sigma} \qquad \Rightarrow \quad B \sim r^{-2}$$
$$h^2 \sim r^{8+4\sigma}$$

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② $$\frac{d}{dt} m(g(t)) = \lim_{r\to\infty} \int_{S_r} \left(\partial_i g_{ij}'(t) - \partial_j g_{ii}'(t)\right) dA^j.$$

$$= -2 \lim_{r\to\infty} \int_{S_r} \underbrace{\left(\partial_i R_{ij}(t) - \partial_j R_{ii}(t)\right)} dA^j.$$

pass these ∂ to ∇.
and apply Bianchi id.

$$= \lim_{r\to\infty} \int_{S_r} (\nabla_j R)(t)\, dA^j. \longrightarrow 0.$$

$$\int_{S_r} |\nabla R|\, dA \le C\left(\int |R|\, dV + \sup |Ric|\right)$$

$$m(g(t))' = 0 \qquad t > 0.$$

idea

- Ricci flow $(M^n, g(t))$ finitely many surgeries.

- then $(M^n, g(t))_{[T,\infty)}$ immortal

  Claim: this implies ~~convergence to~~ global weigthed convergence to $(\mathbb{R}^n, \delta_{ij})$

Analyze $g(t)$ as $t \to \infty$.

μ-functional

$$\overline{W}(g, u, \tau) = \int \left(\tau \left(4|\nabla u|^2 + Ru^2 \right) - u^2 \log u^2 - nu^2 \right) (4\pi\tau)^{-n/2}\, dV.$$

$$\mu(g, \tau) = \inf \left\{ \overline{W}(g, u, \tau) \;\middle|\; u \in W^{1,2}(M), \int (4\pi\tau)^{-n/2} u^2 \, dV = 1 \right\}.$$

monotonicity $\quad \mu(g(t_2), \delta - t_2) \geq \mu(g(t_1), \delta - t_1)$

$\forall \; t_1 \leq t_2 \leq \delta.$

$$\mu(g, \tau) = \mu(\tau^{-1} g, 1)$$

μ well-defined if (M, g) has bounded geometry and $\operatorname{inj}(g) > 0$. and $\exists$ smooth positive minimizer if $\mu_M(g, \tau) < \mu_{M_\infty}(g_\infty, \tau)$

$$(M, g, p_n) \xrightarrow{\text{Cheeger-Gromov}} (M_\infty, g_\infty, p_\infty).$$

For AE. $(M_\infty, g_\infty, p_\infty) = (\mathbb{R}^n, \delta_{ij}, 0)$.

$$\mu_{\mathbb{R}^n}(\delta_{ij}, \tau) = 0 \geq \mu_{M^n}(g, \tau)$$

"=" iff M flat

<u>Thm</u> if (M^n, g) is AE with $R > 0$, then

$$\mu(g, \tau) \xrightarrow{\tau \to \infty} 0.$$

pf. if not, $\exists$ subsequence $\tau_k \to \infty$ but

$$\mu(g, \tau_k) \to \mu_\infty < 0 \quad \text{or} \quad \to -\infty.$$

Euler - Lagrangian equation for u_k

minimize $\overline{W}(g, u, \tau_k)$.

$$\tau_k(-4\Delta u_k + R u_k) - u_k \log u_k^2 - n u_k = \mu(g, \tau_k) u_k.$$

Take $\tilde{u}_k$, g rescale on $\mathbb{R}^n \setminus \overline{B_R(0)}$, as $k \to \infty$.

$\tilde{u}_k \to u_\infty$ solving $-4\Delta u_\infty - u_\infty \log u_\infty^2 - n u_\infty = \mu_\infty u_\infty$

on $\mathbb{R}^n \setminus \{0\}$.

Use u_∞ as a test function to get contradiction.

$$W(\delta, u_\infty, 1) = \underbrace{\int \cdots}_{=0} = \mu_\infty \ (< 0, \text{ we assumed}).$$

$(M, g(t))$

if $0 < \underbrace{\sup |Rm|\, t}_{=C} < \infty$.

we can take (x_i, t_i) s.t. $t_i |Rm|(x_i, t_i) \geq C/2$

$$(M, Q_i g(t_i + Q_i^{-1} t), p_i) \longrightarrow (M_\infty, g_\infty(t), p_\infty)$$

parabolic rescaling on $[0, \infty)$.

$$\mu(g_\infty(0), \tau) \geq \limsup \mu(Q_i g(t_i), \tau)$$

$$= \limsup \mu\left(g(0), \frac{\tau}{Q_i} + t_i\right) = 0$$

$\Rightarrow \mu(g_\infty(0), \tau) = 0 \Rightarrow$ flat

$|Rm| = o(t^{-1})$